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The vicinity of disorder varieties: a systematic expansion

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Abstract. We show that it is possible to obtain information in the vicinity of disorder varieties. This can be achieved through a new and systematic type of diagrammatic expansion, illustrated here on the anisotropic triangular Ising model. Non-perturbative methods can also be used to get an exact expression for some quantities, such as finite field correlation functions, magnetisation and susceptibility restricted to the disorder varieties. This last result confirms an exact result of Dhar and Maillard.

A large number of models exhibit 'disorder varieties' where the partition function can be calculated exactly. First introduced by Stephenson (1970) on the anisotropic Ising model without field, these solutions can be generalised to models where no exact solution is known, such as the 2D anisotropic q-state Potts model with a field (Rujan 1984, Baxter 1984) or 3D Ising models (Welberry and Miller 1978). These solutions provide useful information on the analytical structure of these models, which can be combined with other exact properties or even any available results such as, for instance, high temperature expansions (Enting 1977, 1978).

The knowledge of the partition function restricted to the disorder variety is not the only available information: an infinite number of n-point correlation functions and many other quantities such as the susceptibility can be calculated exactly at the disorder variety.

The simple form of these correlation functions and of different quantities suggests the development of a perturbation theory in the neighbourhood of these special varieties: we develop a new type of diagrammatic expansion, specific to the vicinity of these varieties. For the sake of simplicity the method is illustrated on the example of the 2D anisotropic Ising model (2DAIM) without field on the triangular lattice leading to an exact resummation result.

Let us consider the 2DAIM on the triangular lattice represented in figure 1. The partition function is

$$Z = \sum_{\{\sigma\}} \prod_{\blacktriangle} w \tag{1}$$

where the product is over all hatched triangles, and w is the corresponding Boltzmann

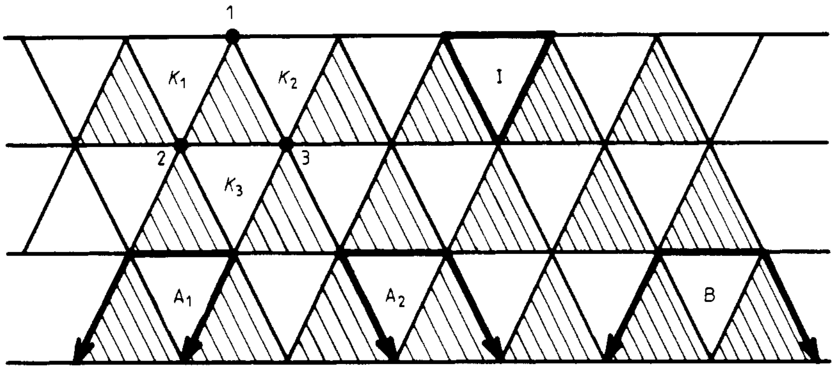

Figure 1. The 2D anisotropic Ising model on the triangular lattice, and the four types of diagrams needed at order T_3.

weight:

$$w = \exp[K_1\sigma_1\sigma_2 + K_2\sigma_1\sigma_3 + K_3\sigma_2\sigma_3 + \tfrac{1}{3}H(\sigma_1+\sigma_2+\sigma_3)]. \tag{2}$$

A disorder variety is known for that model (Jaekel and Maillard 1985):

$$t_1t_2t_3(1+t_1)^2(1+t_2)^2(1-t_3)^2(1-z)^2 + 4(1+t_1t_2t_3)(t_1+t_2t_3)(t_2+t_1t_3)(t_3+t_1t_2)z = 0 \tag{3}$$

where $z = e^{2H}$ and $t_i = \tanh K_i$.

A first method of expansion has been developed by Dhar and Maillard (1985) in zero magnetic field and can be extended in finite field. However it applies only to first derivatives of $\ln Z$. This reduces the calculation of an infinite number of n-point correlation functions to those of a one-dimensional model in finite field: for nearest neighbours see figure 2. The solution of this problem is well known and explicit expressions can be obtained through the transfer matrix method (see, for example, McCoy and Wu 1973). We now show that a knowledge of these finite field correlation functions allows one to obtain the magnetisation on the variety.

Eliminating the field H, the partition function per site is on this variety:

$$\frac{1}{N}\ln Z = \tfrac{1}{2}\ln(1-t_3^2) - \tfrac{1}{2}\ln(1-t_1^2) - \tfrac{1}{2}\ln(1-t_2^2) + \tfrac{1}{2}\ln\left(\frac{t_1t_2}{-t_3}\right) \equiv f(t_1, t_2, t_3). \tag{4}$$

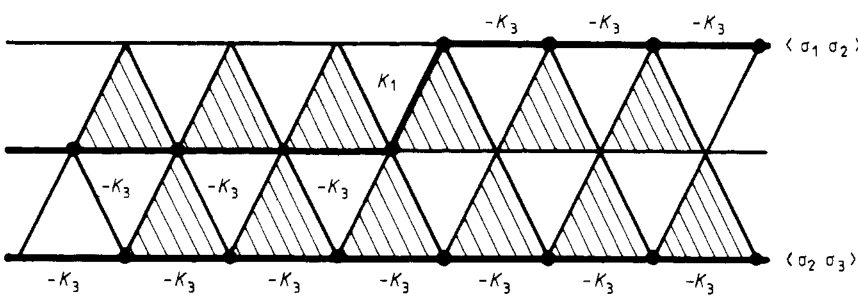

Figure 2. The equivalent one-dimensional models for the calculations of $\langle\sigma_1\sigma_2\rangle$ and $\langle\sigma_2\sigma_3\rangle$, respectively.

This provides an expression for the differential $\mathrm{d}(\ln Z)$ on the variety. On the other hand, the general expression of this differential is:

$$\mathrm{d}(1/N)\ln Z = \langle\sigma_1\sigma_2\rangle\,\mathrm{d}K_1 + \langle\sigma_1\sigma_3\rangle\,\mathrm{d}K_2 + \langle\sigma_2\sigma_3\rangle\,\mathrm{d}K_3 + \tfrac{1}{2}(M/H)\,\mathrm{d}(H^2). \tag{5}$$

One can now use equation (3) in order to express $\mathrm{d}(H^2)$ as a function of the K_i only. Identifying the two resulting expressions of $\mathrm{d}(\ln Z)$ from (15) and (16) one obtains

$$\frac{M}{H} = 2\,\frac{(1-t_1^2)\,\partial f/\partial t_1 - \langle\sigma_1\sigma_2\rangle}{\partial(H^2)/\partial K_1} \tag{6}$$

and similar expressions for directions 2 and 3 which coincide with (6) due to the triangular symmetry between K_1, K_2 and K_3. The explicit expression of M/H is rather tedious and will be given elsewhere. This expression simplifies in the limit $H \to 0$ and gives the magnetic susceptibility restricted to the zero field disorder variety:

$$\chi = \frac{(1+t_1t_2)(1+t_1)(1+t_2)}{(1-t_1t_2)(1-t_1)(1-t_2)} \qquad (\text{on } t_3 + t_1t_2 = 0). \tag{7}$$

This agrees with the triangular limit of the result of Dhar and Maillard for the checkerboard model, which was derived in a completely different way. The straightforward generalisation of this calculation to the checkerboard model has also been performed, leading to an agreement with the result of Dhar and Maillard (1985).

The fact that it is possible to calculate exactly a large number of quantities restricted to the disorder variety and that their expressions become extremely simple in the $H \to 0$ limit suggests that we can develop a new type of diagrammatics in the neighbourhood of these varieties. For that purpose we introduce the variables:

$$c_i = \cosh K_i \tag{8}$$

$$T_i = \frac{t_i + t_jt_k}{1 + t_1t_2t_3} \qquad (i, j, k \text{ all different}).$$

Z can be written:

$$Z = 2^{-N}\lambda^N \sum_{\{\sigma\}} \prod_{\triangle} (1 + T_1\sigma_1\sigma_2 + T_2\sigma_1\sigma_3 + T_3\sigma_2\sigma_3) \tag{9}$$

where

$$\lambda = 2(1 + t_1t_2t_3)c_1c_2c_3 \tag{10}$$

and N is the number of triangles. In these variables, the equations of the three disorder varieties related by the triangular symmetry are given by $T_i = 0$ ($i = 1, 2, 3$, respectively). On these varieties, the partition function per site is simply equal to λ.

In the following, the expansion is performed in the vicinity of $T_3 = 0$. Up to the normalisation factor $(\lambda/2)^N$, expression (9) is nothing but the generating function G of closed paths, connected or not, *which pass at most once on each hatched triangle.* The difference with the usual high temperature expansion lies in that topological constraint. As a consequence, no self-intersections occur. We sketch the calculation of the order T_3 and T_3^2 of G:

$$G = 2^N[1 + T_3Ng_1(T_1, T_2) + T_3^2Ng_2(T_1, T_2) + \ldots] \tag{11}$$

where g_i is the generating function of closed diagrams involving i horizontal links, one of them being specified. Denoting by $N_i(n_1, n_2)$ the number of such diagrams with n_1 (resp. n_2) links of weight T_1 (resp. T_2), one can show that $N_1(n_1, n_2)$ satisfies the following recursion relation:

$$N_1(1,1)=1 \tag{12}$$

$$N_1(n_1, n_2) = N_1(n_1-2, n_2) + N_1(n_1, n_2-2) + \sum_{k_1=1}^{n_1-1} \sum_{k_2=1}^{n_2-1} N_1(k_1, k_2) N_1(n_1-k_1-1, n_2-k_2-1). \tag{13}$$

These relations have a simple geometrical origin, depicted in figure 1. Relation (12) corresponds to the single diagram I. The three terms of relation (13) correspond to diagrams beginning like A_1, A_2 and B, respectively. This leads to the following equation for g_1:

$$T_1 T_2 g_1^2 + (T_1^2 + T_2^2 - 1) g_1 + T_1 T_2 = 0. \tag{14}$$

The only acceptable solution is the one which is regular at the origin. It is also possible, though more involved, to derive geometrically a recursion relation for $N_2(n_1, n_2)$. One then obtains a quadratic equation with matrix coefficients, which will be given in a forthcoming publication.

Let us point out that the diagrammatic expansion of g_1 has been resummed to all orders in T_1 and T_2.

From equations (14) we obtain the expansion of $\ln Z$ up to order T_3^2 in the vicinity of the disorder variety:

$$G_c = (1/N) \ln Z = \ln \lambda + T_3 g_1(T_1, T_2) + T_3^2[g_2(T_1, T_2) - \tfrac{1}{2} N g_1^2] + \ldots. \tag{15}$$

The correlation functions can also be obtained from the expansion (11), since one has the general formula:

$$\langle \sigma_2 \sigma_3 \rangle = T_3 + (1-T_3^2)\partial G_c/\partial T_3 + (T_1 - T_2 T_3)\partial G_c/\partial T_2 + (T_2 - T_1 T_3)\partial G_c/\partial T_1 \tag{16}$$

and two similar formulae for $\langle \sigma_1 \sigma_2 \rangle$ and $\langle \sigma_1 \sigma_3 \rangle$ obtained by permutation.

As a check on our calculation for this exactly solvable model, one can verify that on the variety this reduces to

$$\left.\begin{aligned} \langle \sigma_2 \sigma_3 \rangle &= -t_3 \\ \langle \sigma_1 \sigma_2 \rangle &= t_1 \\ \langle \sigma_1 \sigma_3 \rangle &= t_2 \end{aligned}\right\} \quad \text{at } T_3 = 0 \tag{17}$$

which coincides with the result obtained by the Toeplitz determinant method (Stephenson 1970).

This diagrammatics has also been performed for the checkerboard Ising model without magnetic field. In these exactly solvable cases, this diagrammatics can even be completely resummed, leading to an alternative solution of the 2DAIM. However the expansion given here is independent of this point. All the combinatorial details leading to the recursion equations will be given in a forthcoming publication.

(As a byproduct, we will also study there the properties of the set of diagrams considered as a thermodynamical system of its own, with emphasis on the links with the physics of random walks.)

One can remark that the exact expressions we have obtained in the framework of this perturbative theory are as analytically simple (algebraic expressions) as the expressions we get restricted to these varieties (rational). One would like to generalise similar results to unsolved problems for which the disorder variety does exist, such as Ising models in finite magnetic fields, q-state Potts models and 3D Ising models.

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